The Bell System Technical Journal

January, 1931

The Detection of Two Modulated Waves Which Differ Slightly in Carrier Frequency *

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The present paper contains an analysis of the detection of two waves modulated with the same, or with different, audio frequencies and differing in carrier frequency by several cycles or more. Both parabolic and straight line detectors are treated and there are derived the expressions for all of the important audio frequencies present in the output of these detectors when such waves are impressed. There are discussed the types of interference which result when one station is considerably weaker than the other and simple attenuation formulæ are employed in estimating the character and extent of the interference areas around the two transmitters. Beyond the use of such formulæ no attention is given to phenomena which may occur in the space medium such as fading, diurnal variations in field intensity, etc.

WHENEVER one of two stations operating on the same wavelength assignment wanders from its proper frequency, waves are likely to be received which differ in carrier frequency by several cycles or more. Under such conditions the two signals may be thought of as made up of entirely distinct frequencies and phase relations between analogous components of the two waves need not be considered. In the important case in which the carriers are of identical frequency this is no longer true and phase and its dependence on position and transmission phenomena must be taken into account. This case will be reserved for future study, the present work being limited to a consideration of the phenomena connected with the detection of distinct frequencies.

The most important undesired frequency which is present in the output of the detector is the beat note between the two carriers. It is sometimes carelessly assumed that if the frequency of this beat note is reduced below the audible range the only remaining interference will be due to the speech from the undesired station. Such is not the case and it will be shown later on that when the beat frequency is reduced below the audible range, but not to zero, there remains a group of spurious frequencies which will introduce an interfering background. When the undesired carrier is of relatively small intensity this background is a great deal stronger than the interfering speech. It is therefore desirable to obtain quantitative data on the interfering spec-

* *Proc., I. R. E.*, Jan., 1931.

trum which occurs in the receiver output, in terms of the intensities and degrees of modulation of the input signals.

It is to be expected that the results obtained will depend, to some extent at least, on the type of detector which is used. The square law characteristic is a fair approximation to that of any detecting device which is worked over only a small range and hence an analysis of this characteristic may be expected to serve as an excellent guide to general detector performance. When large signals are impressed on the detector the functioning of the device may approximate more closely to that of the ideal straight line detector. It has been felt that a study of these two types would furnish data from which the performance of any intermediate type of detector could be inferred without great error. As the problem of the square law detector is very much the simpler it will be considered first.

Mathematical Analysis

There will be assumed two broadcasting stations transmitting on frequencies which differ by a relatively small amount, the beat frequency being restricted to the audible range or less. Each of the carriers will be assumed to be modulated by a single audio frequency, the modulating frequencies at the two stations being, in general, different. The total signal impressed on the receiving detector will then be of the form

$$v = E(1 + M \cos pt) \cos \omega_1 t + e(1 + m \cos qt) \cos \omega_2 t, \tag{1}$$

in which

v is the total alternating voltage impressed on the detector.
E is the amplitude of the desired carrier.
e is the amplitude of the undesired carrier.
M is the degree of modulation of the desired signal.
m is the degree of modulation of the undesired signal.
$\omega_1/2\pi$ is the frequency of the desired carrier.
$\omega_2/2\pi$ is the frequency of the undesired carrier.
$p/2\pi$ is the frequency of the desired modulation.
$q/2\pi$ is the frequency of the undesired modulation.

Square Law Detector

We shall first suppose this signal to be impressed on a detector which will be assumed to have a characteristic in the neighborhood of the operating point, of the form

$$i = A_0 + A_1 v + A_2 v^2. \tag{2}$$

An expression of this type will accurately represent a small portion of any continuous characteristic. The present analysis requires that the impressed e.m.f. shall be of small amplitude in order that the limits of the portion of the characteristic thus represented may not be exceeded. This restriction is necessary in treating square law detectors.

The audio frequency output of the detector will be due entirely to the second order term in (2). Hence it will be sufficient, for our purposes, to square the expression for v. We are interested primarily in the ratios of the amplitudes of the various undesired audio frequencies produced to the amplitude of the desired signal of frequency $p/2\pi$. Such a ratio will be designated as a relative amplitude. Neglecting circuit constants, etc., which will apply equally in all the expressions for the various frequencies, the amplitude of the desired component of the audio frequency output is readily shown to be E^2M. The expression for v^2 is reduced to first power sinusoids and the amplitude of each frequency converted to a relative amplitude by dividing by E^2M. The case in hand yields twelve undesired audio frequencies, the relative amplitudes of which are listed in table I. Before commenting on these results we shall consider the straight line detector.

TABLE I

Angular Velocity	Ratio to E^2M	Angular Velocity	Ratio to E^2M
$2p$	$\frac{M}{4}$	$p \pm u$	$\frac{e}{2E}$
q	$\frac{e^2m}{E^2M}$	$q \pm u$	$\frac{em}{2EM}$
$2q$	$\frac{e^2m^2}{4E^2M}$	$p \pm q \pm u$	$\frac{em}{4E}$
u	$\frac{e}{EM}$		

in which $u = \omega_1 - \omega_2$.

The Straight Line Detector

In making analyses of rectification by a straight line detector it is customary to reduce the sum of the various impressed radio frequencies to a single radio frequency, the amplitude and phase angle of which are slow functions of time. The most common example of this type of treatment is a combination of the carrier and two side bands of single frequency modulation into the familiar expression for a modulated

wave in which the amplitude of the radio frequency is an audio frequency function. In this case the radio frequency phase angle is constant. In the case of a single frequency modulation with one side-band eliminated there are impressed on the detector input only two frequencies. These may be combined in a well known manner.[1] Thus, if the impressed voltages are of the form acosx and bcosy, then the amplitude is given by

$$\sqrt{a^2 + b^2 + 2ab \cos (x - y)}. \tag{3}$$

The expression for the phase angle will not be given here as it can be shown that if a and b are unequal and the difference between the frequencies $x/2\pi$ and $y/2\pi$ is small compared with either frequency, then the variation of the phase angle with time may be neglected in computing the audio frequency components. In the present case we have two radio frequency waves the amplitudes of which are not constants but are slow functions of time and these may he substituted for a and b in (3). Thus the effective amplitude of the total input signal may be taken to be

$$S = \sqrt{A^2 + B^2 + 2AB \cos ut}, \tag{4}$$

in which

$$A = E(1 + M \cos pt),$$
$$B = e(1 + m \cos qt),$$

and

$$u = \omega_1 - \omega_2.$$

The problem then resolves itself into an analysis of the detection, by a straight line detector, of a single radio frequency component. The results of such an analysis are well known and it can be readily shown that the audio frequency output may be obtained, except for a factor of proportionality, by resolving the amplitude into its audio frequency components. In the present case the amplitude to be resolved is given by (4) which may be written

$$S = \sqrt{(A + B)^2 - 2AB(1 - \cos ut)}.$$

The interfering signal B will be taken to be always less than the desired signal A, and hence $A^2 + B^2 > 2AB$, from which it follows that $(A + B)^2 > 2AB(1 - \cos ut.)$ Hence the radical may be expanded by the binominal theorem, giving

$$S = A + B - \frac{AB(1 - \cos ut)}{A + B} - \frac{A^2B^2(1 - \cos ut)^2}{2(A + B)^3} - \frac{A^3B^3(1 - \cos ut)^3}{2(A + B)^5} \cdots \tag{5}$$

[1] Vide: Lord Rayleigh, "Theory of Sound," page 23, sec. ed.

It is to be observed that each of the terms of this series, except the first, contains time in the denominator and hence further expansions are necessary. The denominators of the various terms can be expanded by the binominal theorem in such a way as to put all the expressions containing time in the numerators, the expansions being in powers of

$$(ME \cos pt + me \cos qt)/(E + e).$$

By the proper trigonometric transformations it is possible to reduce the final expression for S to frequencies in p, q, u and the sums and differences of the various multiples of these quantities. An additional discussion of this analysis is given in an appendix. In order that the various series involved may converge with a manageable degree of rapidity it is necessary to limit the relative amplitudes of the interfering carriers and the degrees of modulation as well. Consequently the solutions are restricted to intensities of the interfering carrier of 0.1, or less, of the desired carrier and to degrees of modulation of either signal ranging from 0.1 to .5. These limits are suitable also because we are interested chiefly in interference by a relatively weak signal, the interference caused by a signal, the carrier amplitude of which is greater than 0.1 of that of the desired carrier amplitude being near the tolerable limit in the majority of cases. The upper value for the modulation of 0.5 is approximately equal to the average degree of modulation of a station employing as deep modulation as is practical, only the peaks running up to nearly unity. The value of 0.1 for the lower limit is of course transgressed by soft passages in speech or music. However, the range here specified is sufficiently large to give an excellent idea of what may be expected from various degrees of modulation of desired and interfering signals and the results of more extreme cases may be inferred from the data here developed. Under these limits it is found that the only audio frequencies of any importance which appear in the output are:

$$\begin{aligned} S = {} & \left(ME - eg\left(a_0 M - a_1 + a_2 \frac{M}{2} \right) + \frac{m^2 e^2 M g^2}{2E} \right) \cos pt \\ & + \left(me - eg\left(a_0 m - \frac{a_1 M m}{2} - \frac{meg}{E} \right) - \frac{3e^2 g^3 b_0 m}{2E} \right) \cos qt \\ & + \left(\frac{m^2 e^2 g^2}{2E} - \frac{b_0 e^2 g^3 m^2}{4E} \right) \cos 2qt \\ & + \left(eg\left(a_0 - \frac{a_1 M}{2} - \frac{m^2 eg}{2E} \right) + \frac{b_0 e^2 g^3}{2E}(2 + m^2) \right) \cos ut \\ & - \frac{b_0 e^2 g^3}{4E} \cos 2ut \end{aligned}$$

$$+ eg\left(\frac{a_0M}{2} - \frac{a_1}{2} + \frac{a_2M}{4} - \frac{m^2Meg}{4E}\right)\cos\,(p \pm u)t$$

$$+ \left(eg\left(\frac{a_0m}{2} - \frac{a_1Mm}{4} - \frac{meg}{2E}\right) + \frac{b_0e^2g^3m}{E}\right)\cos\,(q \pm u)t. \tag{7}$$

In which

$$\left.\begin{aligned} a_0 &= 1 + \frac{M^2g^2}{2} + \frac{3M^4g^4}{8}, \quad & a_1 &= Mg + \frac{3M^3g^3}{4} + \frac{5M^5g^5}{8}, \\ a_2 &= \frac{M^2g^2}{2} + \frac{M^4g^4}{2}, & b_0 &= 1 + 3M^2g^2, \\ g &= \frac{E}{E+e}. \end{aligned}\right\} \tag{7a}$$

Comparison Between Detectors

It is now possible to make a comparison between the performance of the straight line and the square law detectors. In Figs. 1 to 4 are shown the relative amplitudes of the interfering frequencies in the two cases for various degrees of modulation. The data for the square law case are indicated by dashed lines and for the straight line case by solid lines, and where the two coincide this is noted on the figures. It is to be noted that the expression for the amplitude of the desired frequency $p/2\pi$ is a complicated function. However, computation shows that over the range in which we are interested, the value of this expression does not differ from ME by more than 1 per cent and, therefore, this value has been assumed in computing the relative amplitudes of the other frequencies.

Probably the most striking feature to be noted in comparing the two cases is the similarity of the results. This is particularly evidenced by the carrier beat note of frequency $u/2\pi$ the amplitude of which differs in the two cases by an inappreciable amount. The spurious frequencies $(q \pm u)/2\pi$ also are practically identical for both detectors. There are, however, several important differences as follows:

The group of spurious frequencies of angular velocity $p \pm q \pm u$, which is of appreciable importance in the square law case, is entirely absent from the range of magnitude considered when a straight line detector is employed. The frequencies $(p \pm u)/2\pi$ are greater in the square law case over the range which we have considered, but the curve which represents them has a smaller slope than in the straight line case and for larger values of the interfering signal the intensities of these frequencies would be relatively less with the square law detector. The intensity of the undesired speech q is definitely less in the straight line case than in the square law case but the slope of the q curves is

about the same for both except for $M = m = 0.5$. It is of interest to observe that the interfering speech received on the straight line detector is very much less in intensity than would be the case if the strong desired signal were absent, and that the variation of the amplitude of this frequency with intensity of the undesired carrier is greater when the desired frequency is present. We have here an analytical description of the familiar masking effect which occurs when a strong unmodulated carrier is received simultaneously with a weak modulated signal. For example, when $e/E = 0.1$ it can be seen from Fig. 1 that the relative amplitude of the component of frequency $q/2\pi$ is 0.0063 for the case of the straight line detector. If this component were unaffected by the presence of the strong signal it would have an amplitude proportional to em and a relative amplitude of em/EM which for the values here considered is 0.1. Hence the "masking" effect is here responsible for a reduction of 24 db.

Lastly, it may be mentioned that there are in the case of the straight line detector certain frequencies of small amplitude which are entirely absent from the square law case. However, no frequency is shown the relative amplitude of which is less than 0.01 for all four pairs of values of M and m, as such frequencies are unimportant. An exception is made with regard to $p \pm u$. This is always less than 0.01 over the range considered but is included for the sake of comparison with the square law results.

Further Consideration of Detector Output

The second harmonic of the desired signal is of importance only in the square law case. It is of the nature of a distortion which is independent of the interference and may be omitted from the consideration of the undesired audio frequencies which are a result of the interference. From Figs. 1 to 4 it is evident that the most important interfering frequencies are those of angular velocity, u, $q \pm u$, $p \pm u$ and $p \pm q \pm u$, the last being of importance only in the case of the square law detector. It is with these frequencies, together with that of the interfering speech $q/2\pi$, that we shall be chiefly concerned.

When the relative magnitudes of the interfering frequencies, which are tabulated on page 3, are multiplied by E^2M, the resulting quantities are proportional to the absolute magnitudes of these frequencies. It is to be noted that the frequencies of greatest interest have absolute magnitudes which are linear functions of M or m except $(p \pm q \pm u)/2\pi$ which is proportional to mM, and $u/2\pi$ which is independent of both M and m and will, therefore, be unaffected by the type of modulation employed at either station. In case there are several frequencies

present in the modulation of each station the radio frequency waves will be of the form $E(1 + M_1 \cos p_1 t + M_2 \cos p_2 t + \cdots) \cos \omega_1 t$ and $e(1 + m_1 \cos q_1 t + m_2 \cos q_2 t + \cdots) \cos \omega_2 t$. For every frequency of the former case which contained M as a factor of its amplitude we shall now have several frequencies respectively proportional to M_1, M_2 etc. while an analogous new group will correspond to the former frequencies

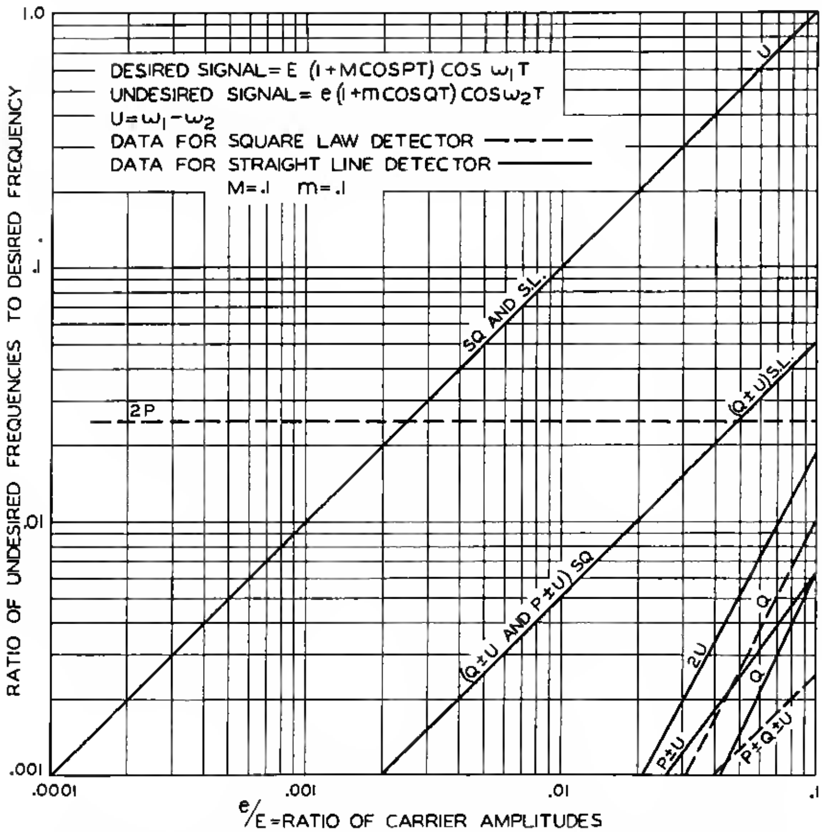

Fig. 1—Relative amplitudes of undesired frequencies as a function of the ratio of the amplitudes of the desired and the interfering carriers. Modulation of both stations small and equal.

containing m. Hence we shall have two frequency spectra derived from the desired speech spectrum containing the p's, but one of the spectra will be shifted upward in frequency by an amount $u/2\pi$ and the other downward by the same amount. Two additional spectra will be derived in a similar manner from the undesired speech spectrum containing the q's. The frequencies of the type $(p \pm q \pm u)/2\pi$ will be

numerous as there will be a product of the M's with each of the m's. However, these are of even moderate importance only when the modulations of both stations are high, and a square law detector is employed at the receiver.

Hence we may picture the interference as made up chiefly of displaced frequency spectra of the type mentioned above, of a carrier

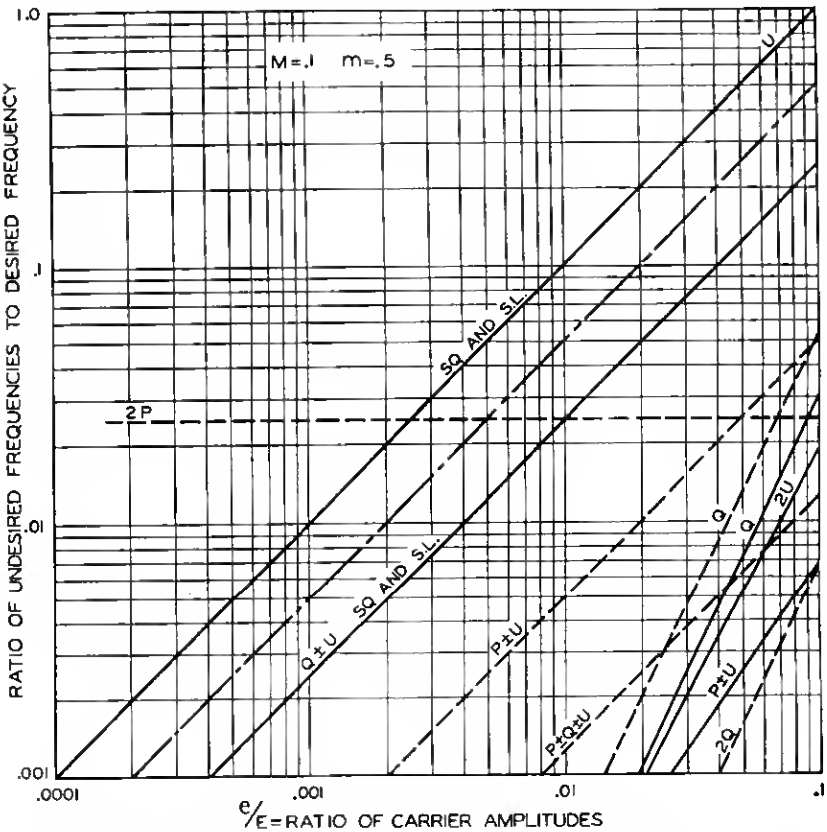

Fig. 2—Relative amplitudes of undesired frequencies as a function of the ratio of the amplitudes of the desired and the interfering carriers. Modulation of desired station small and of interfering station large.

beat and of the interfering speech, which is weak but important because of its intelligibility. The results in the case of a straight line detector would not be very greatly different. The frequencies of the type $(p \pm q \pm u)/2\pi$ would be negligible, the two spectra derived from $p \pm u$ would be much less important and certain new, but rather small cross product frequencies would appear.

In estimating the interference the carrier beat can be considered by itself and from the data at hand there can be derived the areas around each of two stations having approximately the same carrier frequency, inside of which the amplitude of the beat note will be down a given number of db from that of the desired speech. The same is true of the interfering speech when it is different from the desired speech. The

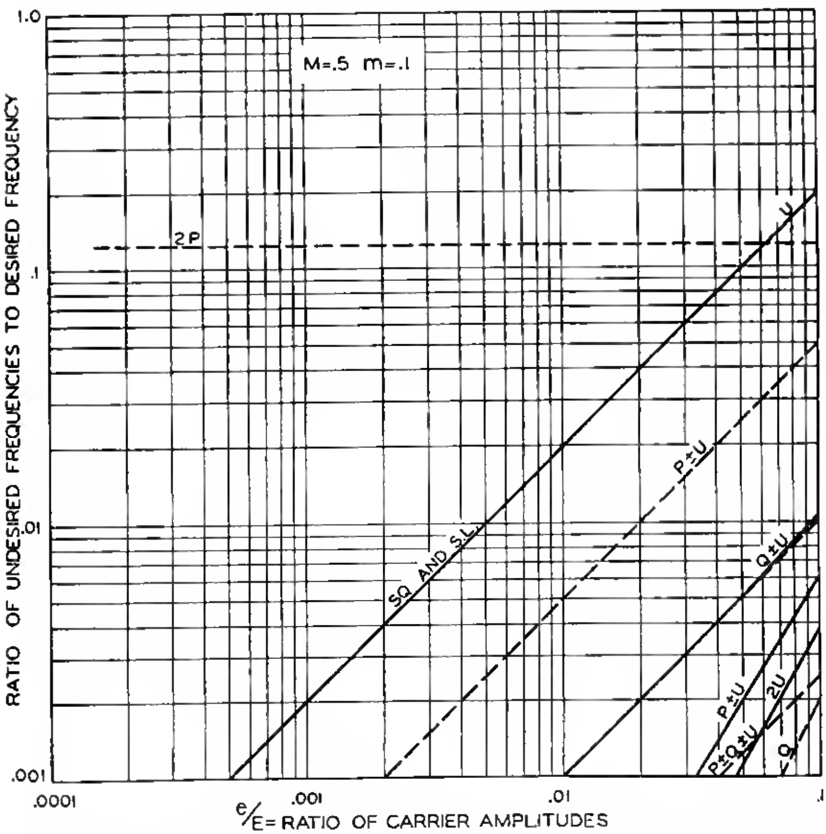

Fig. 3—Relative amplitudes of undesired frequencies as a function of the ratio of the amplitudes of the desired and the interfering carriers. Modulation of desired station large and of interfering station small.

frequencies $(p \pm u)/2\pi$, $(q \pm u)/2\pi$, $(p \pm q \pm u)/2\pi$, etc., will combine to form a disturbing background which we shall designate as "displaced side band interference." This may be taken to include all of the interfering frequencies except those of the undesired speech and its entirely unimportant harmonics. (The frequency $2p/2\pi$ is not here classed as an interfering frequency.)

From Figs. 1 and 4 it is to be noted that when $m = M$ the frequencies $(q \pm u)/2\pi$ are the largest components of the displaced side band interference if a straight line detector is used and have the same amplitude as the $(p \pm u)/2\pi$ components if a square law detector is used. When $m > M$ the $q \pm u$ group is much more important than the $p \pm u$ group as is evident from Fig. 2. When $M > m$ the $q \pm u$

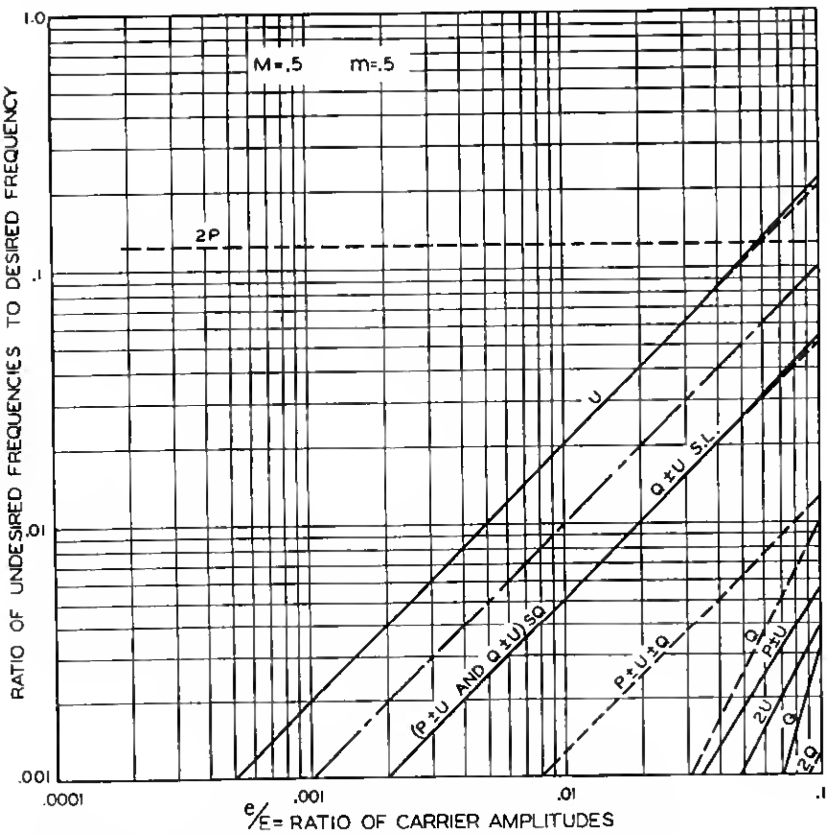

Fig. 4—Relative amplitudes of undesired frequencies as a function of the ratio of the amplitudes of the desired and the interfering carriers. Modulation of both stations large and equal.

group is less important but this case is of no great interest for if the stations are transmitting identical programs, with similar degrees of modulation, it cannot occur and if the programs are different then the interference is determined primarily by what happens when $m > M$. Consequently we may consider that the $q \pm u$ group constitutes the most important part of the displaced side band interference except

when a square law detector is used and the programs are identical. In such a case we shall assume that both stations employ the same degree of modulation and that therefore the $q \pm u$ and $p \pm u$ groups are of the same importance.

Interference Areas of Stations

We have distinguished between three types of interference, namely, carrier beat, unwanted speech and displaced side band. We shall now compute, for several values of attenuation, percentage modulation etc., the areas around a transmitting station inside of which each of these types of interference, due to a second station, will have a relative importance which is not greater than a certain specified amount.

In estimating these areas we must deal with two possible cases which may arise in practice: (1) The two stations transmit different programs. (2) The programs are the same. The carriers are assumed to differ in frequency in both cases.

Case 1

The importance of the various types of interference which are present, will be determined by their ratios to the intensity of the desired speech. In the present case in which the two stations transmit different programs, the amount of interference which may be tolerable will be determined by what occurs when the modulation of the desired station is low, while that of the interfering station is high. Hence, in studying this case we shall make use of Fig. 2, which gives data computed on the basis of a modulation of 0.1 for the desired station and 0.5 for the interfering station.

Taking up first the consideration of the carrier beat note, we shall determine the curve along which the intensity of the beat is down a given number of db from the desired speech. The position of this curve will depend on the degree of modulation of the desired signal, since the lower the modulation the more noticeable will be a beat note of a given intensity. When we have specified the db difference which must exist between these two components of the receiver output the carrier ratio can be picked off from the u line of Fig. 2.

In order to determine the curve along which this carrier ratio exists we shall proceed as follows:

The desired station will be considered to be at the origin of a system of rectangular coordinates and the undesired station will be at the point (D, O). We shall assume that the powers of the desired and undesired stations are P_1 and P_2, respectively, and that their distances from a point in the coordinate plane are d_1 and d_2; then if we denote the

ratio of the carriers by $K = e/E$ the equation of the curve along which the value of K is constant is given by:

$$\frac{K\sqrt{P_1}}{d_1}\epsilon^{-gd_1} = \frac{\sqrt{P_2}}{d_2}\epsilon^{-gd_2}. \tag{8}$$

This equation is based upon a convenient form of the Austin-Cohen [2] formula for the intensity of the field radiated from a radio transmitter. This formula is:

$$E = A\epsilon^{-101.5\alpha d/\lambda^{0.6}}, \tag{9}$$

in which λ is the wave-length in meters, d is the distance from the transmitter in miles and α is an attenuation constant which may range from zero up to 0.01 or even more. In writing down equation (8) we have used the abbreviation:

$$g = \frac{101.5\alpha d}{\lambda^{0.6}}, \tag{10}$$

From (8) there have been computed curves for the case in which $P_1 = P_2$ and for various values of K and α. λ has been taken as 300 meters and D, the distance between the stations, as 1,000 miles.

In Fig. 5 are shown several curves for $\alpha = 0.001$. For small values of K, the curves are practically circular and are of small area. As K increases, the curves become oval shaped and it can be readily shown that for values of K greater than a certain critical amount, the curves will not close but will be of a shape which is roughly hyperbolic.

In Fig. 6 are shown curves corresponding to a value for α of 0.002. It is to be noted that an increase in α enormously increases the area inside of which the ratio of the carriers is less than a certain value. The effect of α will of course be dependent upon the magnitude of the distance between the stations and will be more pronounced the larger this distance. For the present case in which $D = 1{,}000$ miles, there is not much point in considering values of α larger than 0.002, since the attenuation would be so great as to make the effect of one station on the service area of the other of very little consequence.

If we specify that the carrier beat must be at least 40 db down from the speech output due to a 10 per cent modulated signal, then curve 1 of Figs. 5 and 6 will represent the areas inside of which this requirement will be met, while if we call for an interval of 20 db between these two components, curve 5 of Figs. 5 and 6 will represent the areas in which the condition is satisfied. It is evident that if a rigid restriction is placed on the permissible beat note interference which may be allowed, and if the attenuation is of a small value then the area in which the beat

[2] L. W. Austin, *Proc., I. R. E.*, Vol. 14, p. 377.

note may be neglected is extremely small. On the other hand this area increases very rapidly as the attenuation increases.

We may use the same sets of curves in considering the displaced side band interference. From Fig. 2 it is evident that by far the most important components of this interference are those represented by the $(q \pm u)$ group. In order to estimate this interference we must follow some rule for combining the $q + u$ component with the $q - u$ component. In order to do this in a strictly correct manner we should have to take into account the frequencies and sensation levels of the components. However, it has been shown [3] that over a considerable portion of the audio frequency range, and for sensation levels of approximately the magnitude in which we are interested, the interfering effect of these frequencies may be taken to be approximately equal to that due to a single frequency of twice the amplitude of either component. We shall therefore take our data from the dash-dot curve of Fig. 2. From this curve it appears that if the displaced side band interference is to be 40 db down from the desired speech, we must have a carrier ratio of 0.002, while if it is to be 20 db down from the desired speech the corresponding carrier ratio is 0.02. The curves corresponding to these values are shown by 2 and 6, respectively, on Figs. 5 and 6.

From this it appears that the area in which the side band noise is not objectionable may be a great deal larger than that in which the carrier beat is of a tolerable intensity. If the frequency of the carrier beat is reduced below the useful audible range then the former area may be considered to be entirely free from interference of any kind. Consequently, it is highly desirable to limit the maximum possible differences in the carrier frequencies to a value which is definitely below the audio frequency pass band of commercial radio receivers and loud speakers.

Turning now to the undesired speech, we note that it is of very little importance compared with the displaced side band interference. Thus, if this speech is to be 40 db down from the desired speech, the value of the carrier ratio is 0.044 for the case of a square law detector, while for a difference in level of 20 db, the carrier ratio is 0.14. A curve for the case of a 40 db difference is indicated by 7 of Fig. 5.

The comparison between curves 7 and 6 emphasizes the fact that we may have considerable areas of intolerable displaced side band interference in which the intelligible speech from the undesired station is not noticeable. Of course, this interference is often classed as distorted speech but the distinction is convenient in the present discussion.

[3] J. C. Steinberg, "The Relation Between the Loudness of a Sound and its Physical Stimulus," *Phys. Rev.*, Sec. Ser., Vol. 26, pp. 507–523.

Case 2

In this case the programs are identical and consequently the speech from the two stations will undergo simultaneous fluctuations of intensity. We shall here assume that the two stations have the same degree of modulation at any instant. We may then take our data from the curves for which $M = m$. However, this does not apply to the carrier beat note, since its intensity is independent of the degree of

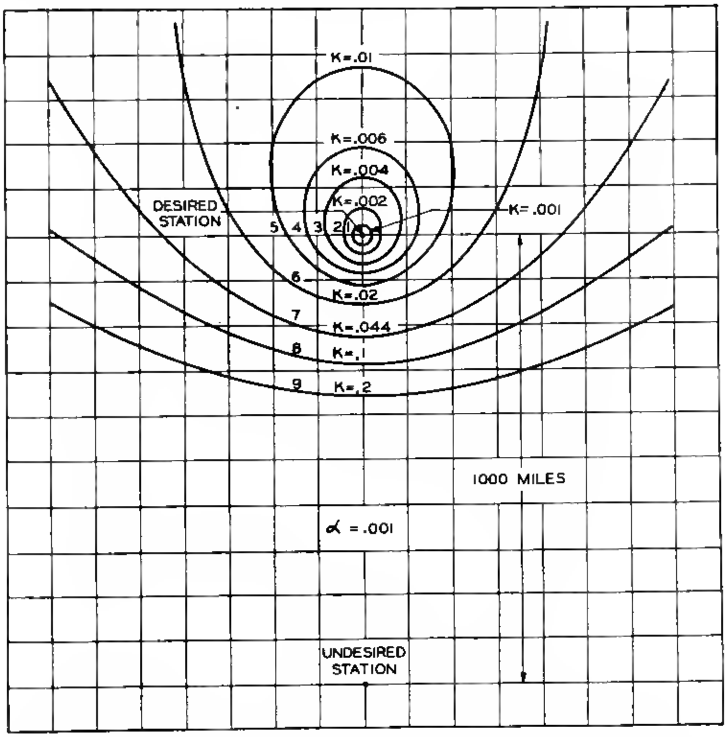

Fig. 5—Curves along which the ratio of the carrier amplitudes received from two stations has a constant value K, as indicated. Attenuation small.

modulation of either station and its interfering effect will be determined by conditions which exist when the desired station has a low degree of modulation. Hence the discussion of this component of the interference will be exactly the same as in the preceding case.

Referring to Figs. 1 and 4, it is evident that by far the greatest portion of the displaced side band interference is due to the $q \pm u$ components, in the case of the straight line detector, and the $q \pm u$ and $p \pm u$ components in the case of the square law detector. The

identity of the curves for these components in the two figures shows that the degree of modulation has practically no effect on the relative importance of the interference which occurs when the same programs are transmitted.

If we again assume that the total interference may be represented by a fictitious component of twice the amplitude of the $q + u$ component,

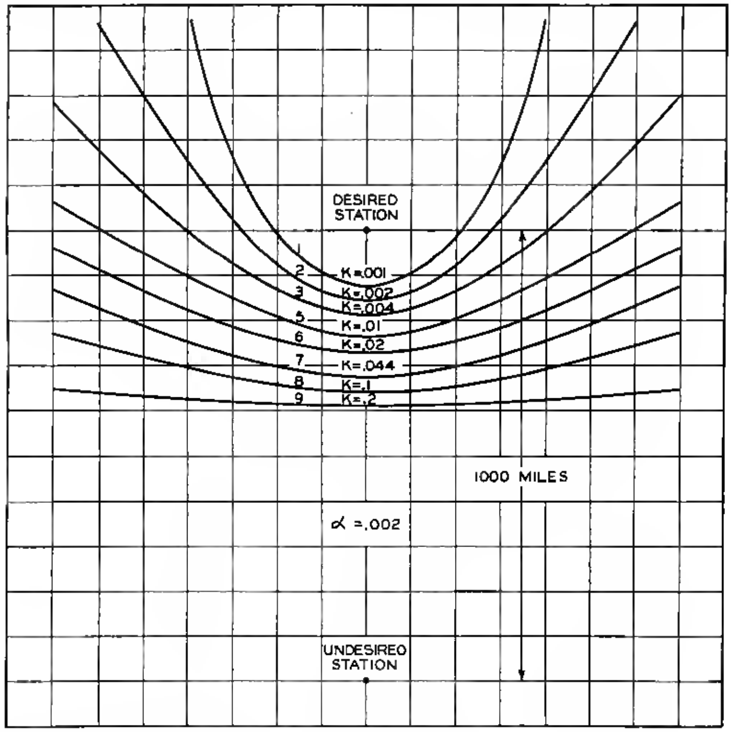

Fig. 6—Relative amplitudes of undesired frequencies as a function of the ratio of the amplitudes of the desired and the interfering carriers. Attenuation constant α twice that of Fig. 5.

we may take our data from the dash-dot line of Fig. 4. This should represent the case fairly well for the straight line detector but when a square law detector is used, greater interference should result due to the importance of the $p \pm u$ terms. However, we shall consider only the $q \pm u$ group and the phenomena associated with the square law case may be readily inferred. In order that the displaced side band interference may be 40 db down from the desired speech the carrier ratio must have a value of 0.01, while if it is to be 20 db down, this value must be 0.1. The first value corresponds to curves 5 of Figs.

5 and 6, while the second value corresponds to curves 8. We observe that there is a tremendous difference between the areas which may be considered to be free from displaced side band interference and those which will be free from carrier beat interference, in case the beat frequency is allowed to wander into the audible range. The comparison between the two areas is given by curves 1 and 5 for the 40 db interval and by curves 5 and 8 for the 20 db interval.

The speech from the interfering station will now be the same as the desired speech and can have effect only in so far as it adds to or subtracts from the desired speech. It will be noted from Figs. 1 and 5 that for carrier ratios of less than 0.1 this component is always down more than 40 db and may be safely neglected.

The foregoing discussion serves to illustrate the types of interference which may be expected when two stations are operated on approximately the same frequency. The data discussed have involved low values of attenuation. This is of particular interest when the distance between stations is large since with high values of attenuation either station will have very little effect on the service area of the other. Of course at night time we may have signal strengths which will be of the order of magnitude of that given by the simple inverse distance law involving zero attenuation. This possibility probably presents a serious limitation on night time common frequency broadcasting but should be of little consequence during the daylight hours. Conditions will be somewhat different for stations that are placed nearer together and specific results can be readily computed for any given spacing. The equations which have been discussed can be applied to any such case and the areas corresponding to those in Figs. 5 and 6 determined.

One point which is emphasized by the results which have been obtained is, that with a carrier frequency difference of several cycles satisfactory reception cannot be expected in the regions which lie midway between two transmitters. The field strength of one station must be at all times predominately higher than that of the other and consequently the use of pseudocommon frequency broadcasting should be restricted to stations of wide geographic separation. It should then be possible to furnish high grade service to relatively small densely populated areas in the immediate vicinity of either transmitter, reception at a considerable distance from both stations being admittedly unsatisfactory. However, if the carriers are strictly isochronous much larger service areas should be feasible.

APPENDIX

Equation (5) is

$$S = \overset{\text{I}}{A + B} - \overset{\text{II}}{\frac{AB(1 - \cos ut)}{A + B}} - \overset{\text{III}}{\frac{A^2B^2(1 - \cos ut)^2}{2(A + B)^3}} - \overset{\text{IV}}{\frac{A^3B^3(1 - \cos ut)^3}{2(A + B)^5}}. \qquad (5)$$

To expand these terms we write

$$\frac{1}{(A+B)^n} = \frac{1}{(E+e+ME\cos pt+me\cos qt)^n}$$
$$= \frac{1}{(E+e)^n}\left(1 - \frac{n(ME\cos pt+me\cos qt)}{E+e} + \frac{n(n+1)(ME\cos pt+me\cos qt)^2}{2(E+e)^2}\cdots\right.$$
$$\left. +(-1)^r\frac{n(n+1)(n+2)\cdots(n+r-1)(ME\cos pt+me\cos qt)^r}{\underline{|r}(E+e)^r}\right). \qquad (5a)$$

It is evident there are present in S an infinite number of frequencies and it is necessary to select those which are of appreciable magnitude relative to that of the desired frequency of amplitude EM. Fortunately these are not very numerous.

In deciding whether or not a given term should be retained there are two points to be considered: (1) whether all the terms of a given frequency total to a value sufficiently large to call for the presence of this term in the final result; (2) what per cent accuracy should be required in the frequencies which are retained. Thus if it is desired to retain all frequencies the relative amplitude of which is greater than 0.01 we cannot arbitrarily retain all individual terms which make a contribution of 0.01 or greater and neglect those of relative importance of less than 0.01. Thus if a term of a given frequency has a relative amplitude of 0.01 and another term of the same frequency a relative amplitude of 0.009 the second term should be retained. Otherwise we should have a large percentage error in the value of the amplitude of this frequency. On the other hand it is not desirable to maintain the same degree of accuracy for the case of retained frequencies of slight relative importance as for those of large importance. As a compromise all individual terms have been retained which, after division by EM, are of a magnitude greater than 0.005 for any values of M, m and e/E which are here dealt with. An exception is made in

the case of a term in cos pt derived from term III of (5). This term is slightly larger than the above limit when $M = 0.5$ and $e/E = 0.1$ but as it decreases rapidly with a decrease in e/E it has been omitted for the sake of simplicity.

Having chosen this limit of 0.005 for the relative magnitude of individual terms it can be shown to be permissible to neglect term IV and all subsequent terms of (5). Furthermore, only a few of the large number of terms yielded by III need be retained.

After applying these rules there appear several frequencies that are never as large as 0.01 in relative magnitude and these have been omitted from consideration. As has been stated in the body of the paper, an exception is made in the case of the frequencies $(p \pm q \pm u)/2\pi$. If a given frequency exceeds 0.01 for any one of the four pairs of values of M and m, it has been shown on the figures for all of the pairs.

After the formula (5a) has been applied to S and the expressions for A and B inserted there remains the necessity of reducing products and powers of various sinusoidal terms to sums of simple first order sinusoids. This is a tedious procedure but is a matter of simple trigonometry and will not be set forth in detail.

From (5a) it can be seen that if M or m is near unity the series will converge very slowly. Furthermore, since to obtain relative magnitudes we divide by M, it is impossible to obtain satisfactory convergence due to small values of M in the denominator. Hence it is necessary to limit M and m to 0.5 or less and in addition M must be no smaller than 0.1. It would be permissible to allow m to become less than 0.1 but as little would be gained by this m has been restricted to the same range as M.